

Global warming in mathematical model of multifractal dynamics

*A. N. Kudinov, O. I. Krylova, V. P. Tsvetkov,
I. V. Tsvetkov*

Tver State University, Tver, Russia

Abstract. In this work the variations of global temperature that have occurred in the period from 1860 up to now are analyzed on the basis of the concept of multifractal dynamics. The multifractal curve describing dynamics of global temperature for this period of time has the following values of fractal dimensions over 5 periods lasting for 30–31 years each, accordingly: $D_1 = 1.140$; $D_2 = 1.166$; $D_3 = 1.141$; $D_4 = 1.203$; $D_5 = 1.183$. Such relatively small values of fractal dimensions are indicative of essentially determined character of processes responsible for variations of global temperature. Our predictive estimates provide 0.5°C increase in global temperature by 2072, thereby confirming maintenance of the tendency of global warming in the near future.

Introduction

The meteorological data for the period from 1850 till now, let *Jones and Wigley* [1990] to deduce that the planet climate had warmed, and this change in temperature is 0.5°C . In the nature of things, the question now arises of whether this conclusion is true. The more intricate question is the one of possibility of further warming. There are a lot of facts which undoubtedly have an effect on measurement data thereby causing an apparent warming effect.

In the circumstances concerned, the mathematical climate modeling assumes great importance.

There are no any models in view which describe with the specified degree of accuracy complex atmospheric and ocean physics, which can prove that greenhouse gas emission essentially affects the Earth's climate fluctuation. In whole, temperature increase is in keeping with the previous industrial revolution in consequence of which it has been established that the concentration of carbon dioxide and other greenhouse gases was essentially increased in the atmosphere.

About fifty years ago the attempts to determine the Earth temperature trends were taken. But initially, it was impossible because of the fewness of the view

points. Since 1850 all National Weather Services have been concordantly collecting and maintaining temperature data. Gradually, the weather surveillance network extended over the world, and by the end of 1950 it covered Antarctica. However, even at the present time in some regions especially in the ocean areas having rugged depth contour, measurements are taken seldom enough. But the partial covering doesn't constitute a serious problem; it is compensated by satellite observations.

Notwithstanding the fact that the observational data denote the temperature rise over the last 120 years, there are a lot of questions to decide. That is, how considerable the warming trend is? What is the reason for it? Is it connected with the greenhouse effect?

The key factors affecting the average annual temperature variations

The majority of year to year variations of climate are connected with the internal factors including the atmosphere circulation change. For longer spaces of 2 - 8 years the climate variations are generally deter-

mined from the vertical convection current changes in the ocean and from the ocean surface temperature. The slow temperature response of the oceans results in considerable climatic variations for decades and for a longer time [*Jones et al.*, 1990]. In climate modeling the temperature inertia effect of the oceans is taken into consideration as a random noise representing observable high-frequency year to year variations of average global temperature.

About a half value of warming observed last century we can assign to natural internal variations as throughout the duration of this century the resulting low-frequency variations of temperature have reached 0.2–0.3°C. There is a counter point of view such as the warming value was as high as 0.7–0.8°C, but it was considerably reduced by decreasing of temperature because of internal factors [*Jones et al.*, 1990].

There are various external factors affecting the global climate. One of these factors is a solar fluctuation. Satellite observations show the solar radiant flux varies within 0.1% over the 11-year solar cycle. It corresponds to altering the radiation value falling into the stratosphere within 0.24 W m^{-2} .

The climate doesn't respond these changes immediately. The temperature inertia of the oceans prevents

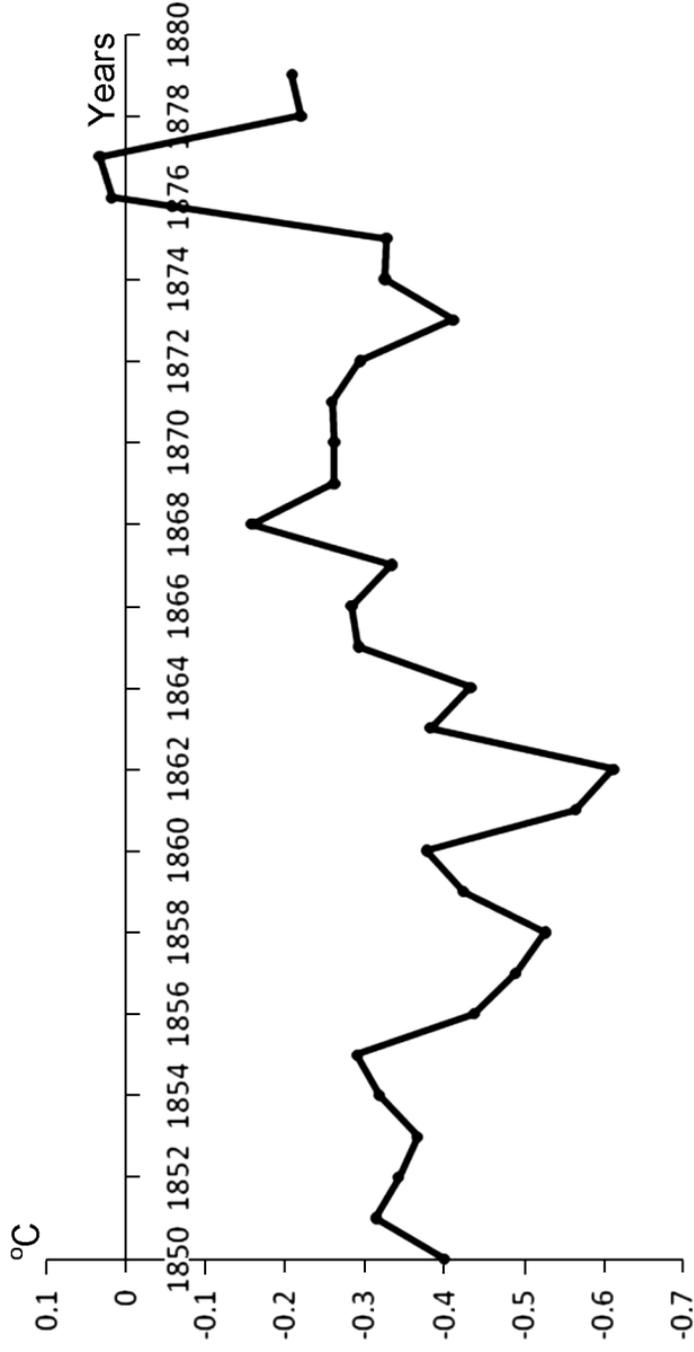


Figure 1. Graph of yearly average temperature variations over the period of 30 years.

the climate from responding quickly. At that, the actual temperature variations will not exceed 0.03°C . For longer time intervals the solar radiation value could alter considerably. The interval between mass advances of glaciers correlates to extended periods of low activity of the Sun such as Maunder's, Isterer's, and Wolff's minimums in 1645–1715, 1450–1550, and 1280–1350, correspondingly.

Anyway, there have not been founded any extended periods of anomalous solar activity since the event of Maunder's minimum. At the present time the effect of solar radiation on the year to year variations of average global temperature on a scale of a century is indefinite but subject to be solved by means of extant and projected satellite measurements.

As for the internal factors, before everything else, we can point out a volcanic activity as a factor affecting the climate. The volcanic explosions resulting in injection of great quantity of dust and sulfates into the stratosphere can cause a substantial fall of temperature in minimal time. In several months after eruption, the temperature condition becomes having repercussions. This temperature effect at the measured level can be observed within the space of two years. For the time being, it is difficult to draw precise conclusions of

the effect of volcanic activity on long-period climatic changes.

As was mentioned above the greenhouse effect can play a heavy role in the case of global warming. There is solid data for greenhouse gas concentration throughout the past few decades. Since 1765 the ambient carbon dioxide concentration has increased from 280×10^{-9} to 350×10^{-9} , the methane concentration – from 800×10^{-12} to 1700×10^{-12} , and the nitrogen oxide concentration has increased from 285×10^{-9} to 310×10^{-9} . The chlorfluorcarbon concentration for some time past (about 40 years), has increased from zero value to 10^{-12} [*Jones et al.*, 1990].

The estimates show that change in greenhouse gas concentration results in change in radiation balance equivalent to solar radiation enhancement by 1%. Consequently, the elevated concentration of greenhouse gases can result in global temperature increase by 0.8–2.6°C. Yet by virtue of the temperature inertia of the ocean, the global warming effect is decaying considerably. It shows the value of 0.5–1.3°C. Consequently, the observed warming of 0.5°C is at the limit of compliance with a possible greenhouse gas effect [*Jones et al.*, 1990].

Many uncertainties in diagnosis of causes of warming

can't be eliminated by virtue of the fact that there are no necessary archival data. It is impossible to give an unambiguous interpretation to global warming over the past century.

The resolution of available uncertainty is possible by means of enhancement of the models valuating the greenhouse effect contribution and giving more correct prediction of variations of climate which are to come.

The work objective is formulation of a mathematical model independent of the particular composition of the system responsible for global temperature variations. This model is based on the multifractal dynamics model [*Mandelbrot, 1982*], [*Kudinov et al., 2011*] offering an opportunity to describe the processes with the parameter time dependency reported in terms of multifractal curves.

Self-similarity of global temperature variations

The first person who turned his attention to a self-similar behavior pattern of dynamic characteristics of systems was *Benoit Mandelbrot* [1982], an originator of fractals.

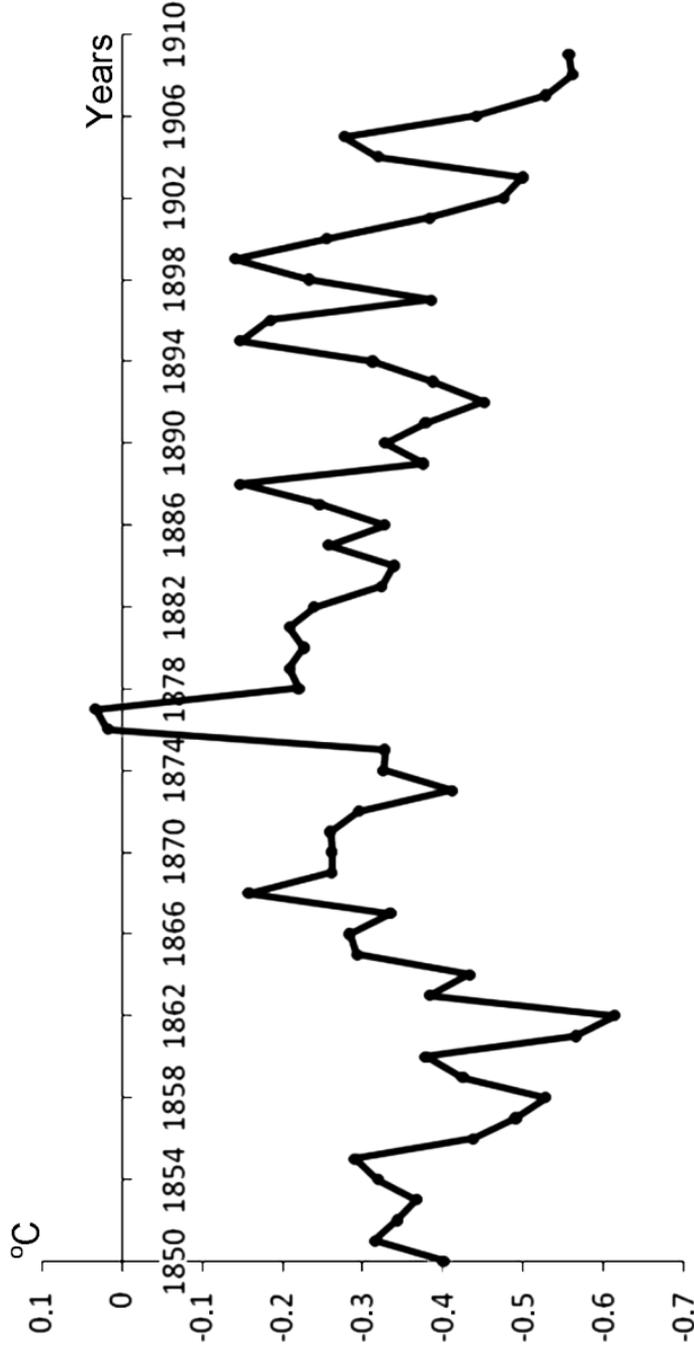


Figure 2. Graph of yearly average temperature variations over the period of 60 years.

Mandelbrot found out that arbitrary external cotton price variations can follow the hidden time-dependant mathematical order which can't be described by standard curves.

Benoit Mandelbrot addressed himself to the question of cotton price statistics observed over a long period of time – there was reliable data on these prices for more than a century. The price variations throughout the day seemed to be unpredictable but the computer analysis could trace the price tendency. The analysis developed a graph showing the price variations over the day certain superimposed on a longer period of time. Mandelbrot traced the symmetry in long-period and short-period price variations. This finding turned out to be a complete unexpectedness for economists who had used mathematics for calculations only. And Mandelbrot himself was surprised at his own findings. Later it was found that he intuitively had begun devising a recursive (fractal) technique in the economic field. The more specific technical term for similarity between the parts and the whole is “self-affinity”. This term is connected with the big-name concept of fractality called “self-similarity” in which every detail of a picture is scaled down or enlarged in the equal ratio.

We shall illustrate a self-similar rate of yearly average global temperature curves graphically.

Let us construct the global temperature history graphs covering the periods of 30 years (1850–1880), of 60 years (1850–1910), and of 155 years (1850–2005) by applying the data from [Brohan et al., 2006]. These graphs are given in Figure 1, Figure 2, and Figure 3.

As is clear from the figures, the rates of the graphs do not change in spite of the fact that every time the scale is changing about twice-three times as much.

Mathematical model of multifractal dynamics and the catastrophes of this model

We shall summarize the elements of multifractal dynamics [Kudinov et al., 2011].

Definition: Let $y(t)$ be a multifractal curve describing the dynamics of the value we are interested in and having the defined fractal dimension value of D_i on the intervals of time T_i ($i = 1, 2, 3 \dots n$).

If the rate X_i of the linear trend $\bar{y}_i(t)$ approximating this function on the interval T_i with the required degree

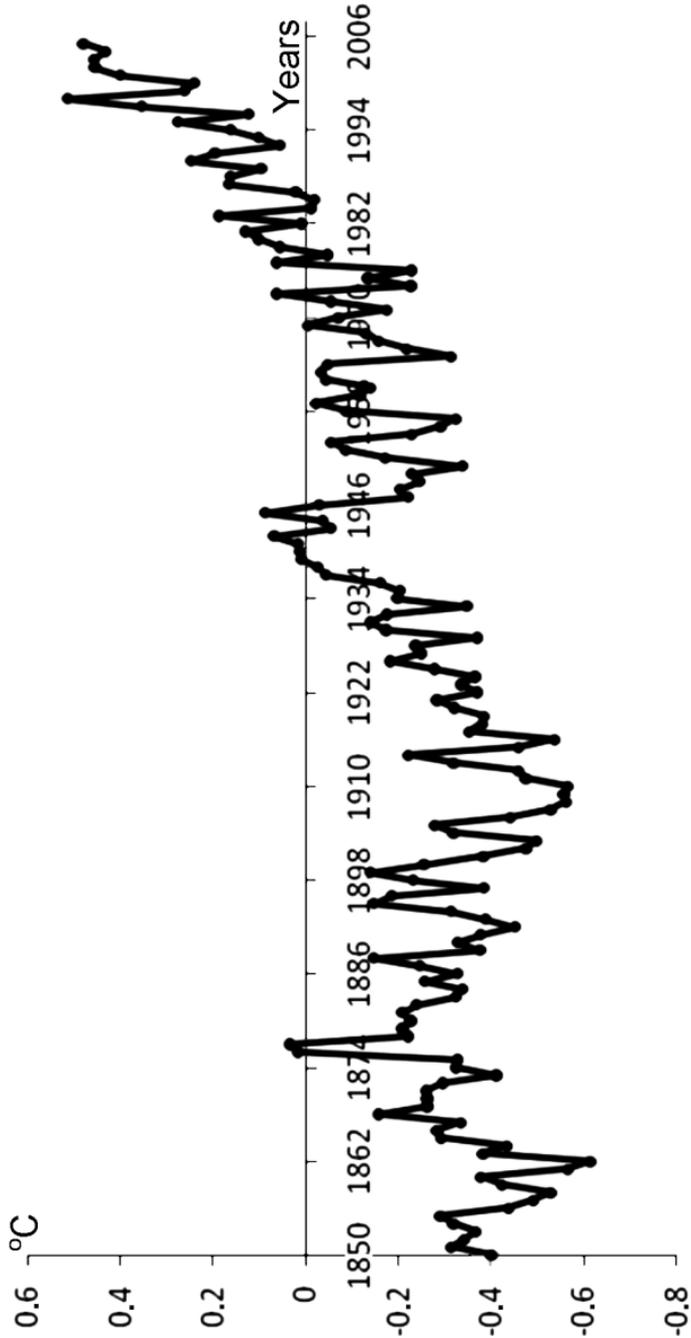


Figure 3. Graph of yearly average temperature variations over the period of 155 years.

of accuracy, is only depends on D_i ; then the given type of dynamics we will designate as multifractal.

In this case in [Kudinov et al., 2011] we offer the following approach: the multifractal process dynamics on the intervals T_i ($t_{0i} < t < t_{0i+1}$, $T_i = t_{0i+1} - t_{0i}$) can be divided into two components by means of using the idea of the linear trend

$$y_i(t) = \bar{y}_i(t) + \tilde{y}_i(t), \quad (1)$$

where $\bar{y}_i(t)$ is a linear trend of the process varying with time smoothly; $\tilde{y}_i(t)(t)$ - are fast oscillations with respect to a trend. It is suggested that $|\bar{y}_i(t)| \gg |\tilde{y}_i(t)(t)|$, and the curve $y_i(t)$ is a multifractal one. The trend line $\bar{y}_i(t)$ has a unit fractal dimension, and $\tilde{y}_i(t)$ has a fractal dimension of D_i .

As the measure of error for the model we will take a value of $\Delta_i = \max |\tilde{y}_i(t)|$ on the run under review D_i . Over the whole run under review the error common value is as $\Delta = \max \Delta_i$, $i = 1 \dots n$.

It is suggested in the multifractal dynamics model that the tangent of the linear trend angle $\bar{y}(t)$ is a function of the fractal dimension D

$$\tilde{y}_i(t) = X_i(D_i)(t - t_{0i}).$$

In our situation, under the character of $y(t)$ we can see a global average annual temperature.

The significant instant of the approach [Kudinov et al., 2011] is a possibility of describing of catastrophes within its limits.

In the segments of the multifractal curve with a constant value of D the slope ratio of the linear trend (the average velocity of the corresponding process) according to [Kudinov et al., 2011], is a function of D and to be determined from the solution of the cubic equation

$$A(D)X + B_k X^3 = \eta. \quad (2)$$

It's conveniently to choose a scale enabling to meet the condition $|X| \ll 1$. The parameter η describes an effective action of external factors on the system under investigation.

For the function $A(D)$, let us choose the following analytic representation [Kudinov et al., 2011]

$$A(D) = \begin{cases} (D_0 - D)^{-1} & \text{if } 1 \leq D \leq D_0, \\ (D_0 - D_k)^{-1}(D_0 - D)^{-1}(D - D_k) & \text{if } D_0 \leq D \leq 2 \end{cases}. \quad (3)$$

The formula (3) offers to describe the variety of behaviors of the linear trend $X(D)$.

The model parameters D_0 , D_k , B_k and η are to be selected from the best possible fit with experimental results.

In case that $D < D_k$, we can neglect B_k member; in this case the following linear approximation is true

$$X = \eta(D_0 - D). \quad (4)$$

In the range of values of D the equation (2) has one real root determined by formula (2).

When D goes to D_k , the situation is altering essentially, and we can't neglect B_k member in (2) in the circumstances.

The equation (2) can be obtained as extreme points of the Fractal Determining Function (FDF):

$$V(X) = \frac{1}{4}X^4 + \frac{a}{2}X^2 + bX \quad (5)$$

The factor $\frac{1}{4}$ is chosen for reasons of convenience. The control parameters in (5) will be a and b , correspondingly.

The extreme points of (5) are to be defined from the following conditions: $\frac{\partial V}{\partial X} = 0$. As a result, we will obtain the following equation:

$$X^3 + aX + b = 0 \quad (6)$$

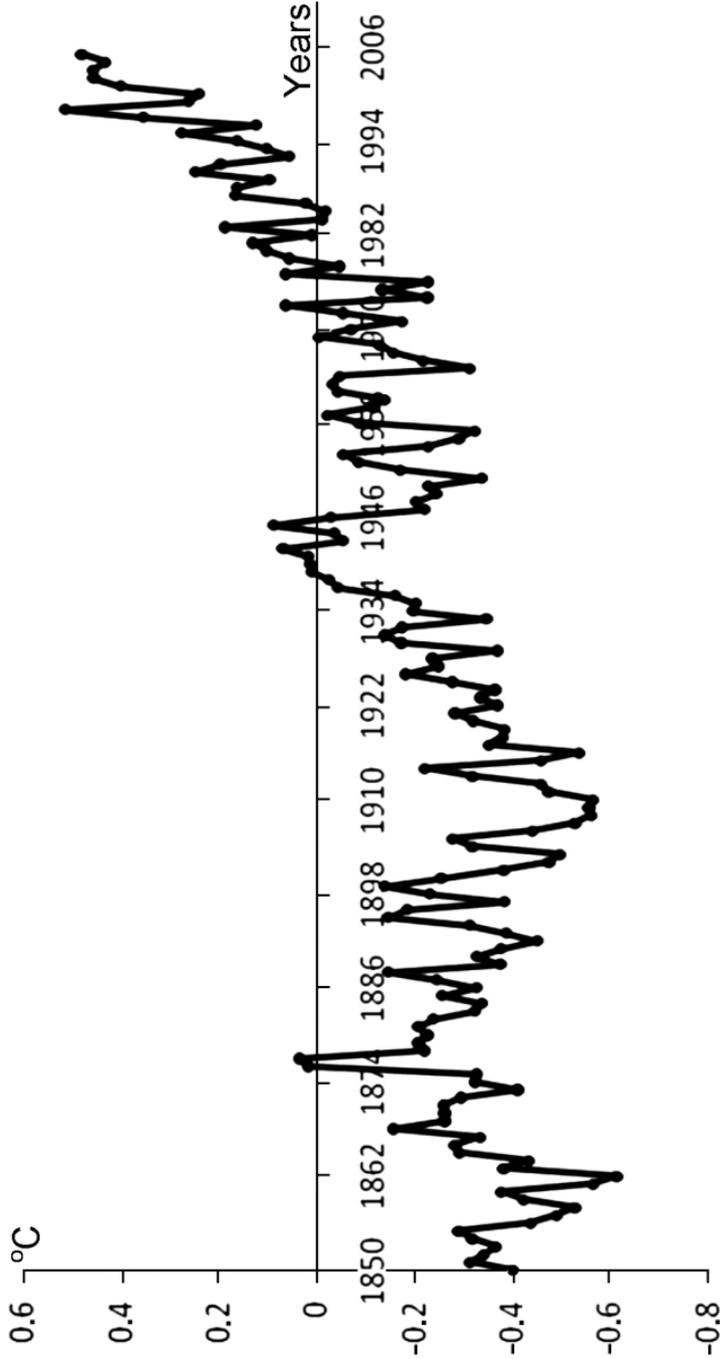


Figure 4. Graph of yearly average temperature variations over the period from 1850 to 2005.

The critical point X_k corresponds to $a = 0$. In case of $b \neq 0$, the analytic view of $X_k(a, b)$ is given in [Kudinov et al., 2009].

In (5) the catastrophe germ is equal to $(1/4)X^4$, and this implies that the catastrophe of A_3 type exists in this model, if $a = 0$, $b = 0$. Moreover, if b parameter depends on η ($b = -\eta/B_k$) parameter that a will be a complicated function of the parameters D_0 , D_k , B_k , D of the fractal model. From (3) we can obtain

$$a(D) = \begin{cases} B_k^{-1}(D_0 - D)^{-1} & \text{if } 1 \leq D \leq D_0 \\ B_k^{-1}(D_0 - D_k)^{-1}(D_0 - D)^{-1}(D - D_k) & \text{if } D_0 \leq D \leq 2 \end{cases} \quad (7)$$

From (7) this implies that the catastrophe of A_3 takes place if $D = D_k$ and $\eta = 0$. In this case the separatrix equation has the following view:

$$\eta = \pm \frac{2}{\sqrt{27}} \left(-\frac{A(D)}{B_k} \right)^{\frac{3}{2}} \quad (8)$$

Table 1.

| i | 1 | 2 | 3 | 4 | 5 |
|---|--------|---------|--------|---------|--------|
| $T_i, \text{ year}$ | 30 | 31 | 30 | 31 | 30 |
| $X_i, \frac{^{\circ}\text{C}}{\text{year}}$ | 0,0089 | -0,0081 | 0,0135 | -0,0016 | 0,0183 |
| D_i | 1,140 | 1,166 | 1,141 | 1,203 | 1,183 |
| Δ_i | 0,2624 | 0,229 | 0,1987 | 0,2281 | 0,1976 |

Global temperature analysis in the multifractal dynamics model with consideration for a linear trend

The data observed for past 160 years show [*Brohan et al., 2006*] that the global temperature T_oC oscillates near the equilibrium value. On this basis, let us make a replacement

$$T = u + T_0.$$

The global temperature curve u showing the temperature from 1850 is given in Figure 4, on a basis of data from [*Brohan et al., 2006*].

Based on the multifractal dynamics model [*Kudinov et al., 2011*], let us classify the total time interval (160

years) under 5 segments T_i ($i = 1, 2, 3, 4, 5$), and perform a linear trend approximation of $u \bar{u}$ in each of them, i.e.

$$u_i = \bar{u}_{0i} + X_i(D_i)(t - t_{0i}) + \tilde{u}_i = \bar{u}_i + \tilde{u}_i \quad (9)$$

Function $X_i(D_i)$ should satisfy the equation (2). The value $\Delta_i = \max |\tilde{u}_i|$ is an approximation error.

The computational results of Δ_i , D_i , X_i are given in Table 1, and the linear trend approximation is shown in Figure 5.

The experimental results are in keeping with the multifractal dynamics model [Kudinov et al., 2011] well enough if for the first three periods $i = 1, 2, 3$ there have been chosen $D_0 = 1.157$ and $\eta = 0.862 \text{ } ^\circ\text{C yr}^{-1}$, and over the last two periods $i = 5, 6$ there have been chosen $D_0 = 1.201$ and $\eta = 0.995 \text{ } ^\circ\text{C yr}^{-1}$. The difference can be noticeable for X_1 only and it will be approximately 30%. This may be due to all manner of inaccuracies of global temperature measurements in the initial period of instrumental measurements. Some increase of η factor in 4 and 5 periods, in comparison with 1, 2, and 3 periods is valid, and this event characterizes the Sun activity and the dynamic activity of the atmosphere of the Earth providing for redistribution of resulting solar energy.

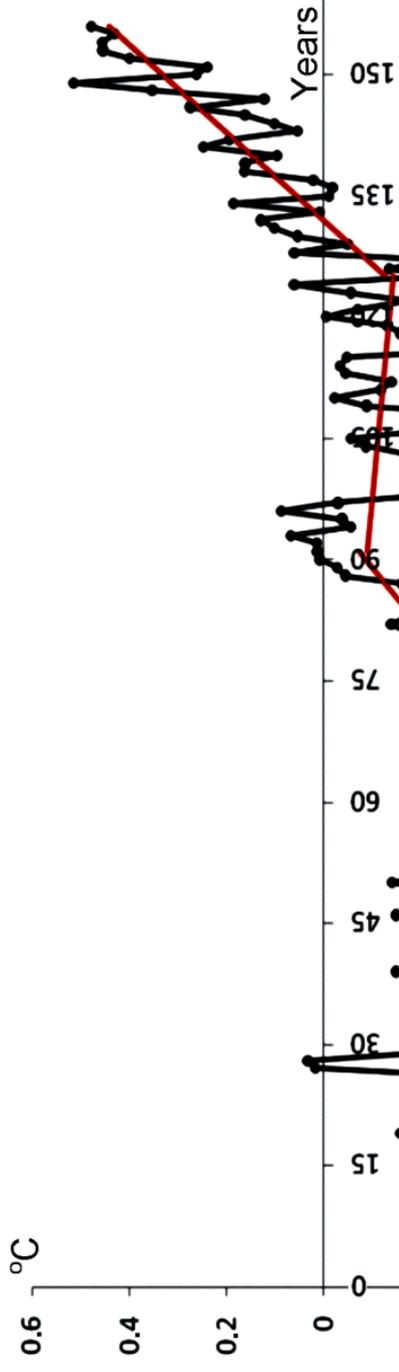


Figure 5. Graph of linear trend approximation of yearly average temperature variations.

Table 2.

| i | 1 | 2 | 3 | 4 | 5 |
|---|--------|-------|--------|-------|--------|
| $X, \frac{^{\circ}\text{C}}{\text{year}}$ | -0,017 | 0,021 | -0,015 | 0,020 | -0,017 |

From the existing experimental results, the critical value of D_k whereby $A(D_k) = 0$ cannot be defined. Its value is assumed to be known like in the financial processes [Kudinov *et al.*, 2009] belonging to $1.67 \leq D_k \leq 1.75$ interval. By virtue of the fact that all D_i values are notably less than D_k , we can suggest there should not be any catastrophes in the global temperature dynamics in accordance with the multifractal dynamics model. There is no reason to wait for their appearance now.

The sufficiently low values of $D_i \leq 1.201$ fractal dimensions, in comparison with Gaussian value of 1.5 suggest the essentially determined nature of processes responsible for the global temperature dynamics. The rise of η factor, $\Delta\eta = 0.133 \text{ }^{\circ}\text{C yr}^{-1}$, in 1950 in our model counts in favor of a global warming trend at the present day.

A slight growth of D_0 from 1.157 to 1.201 over the same period, gives evidence of a slight increase of chaotization of global temperature.

Considering that the global temperature dynamics process is an oscillating one, in the adjacent time intervals T_i the velocity values of X_i linear trends change sign, in other words, there is good reason to believe that for the next interval $T_6 \approx 31$ year we will have $X_6 < 0$. This points to the possibility of reduction of the global temperature trend in this period. The estimated specific value of reduction rate of u we will present in the following section.

Global temperature analysis in the multifractal dynamics model with consideration for a nonlinear trend

This section integrates the results obtained in earlier sections for a nonlinear trend. This will permit the trend function to be not only continuous throughout the observation interval but a differentiable one. It should be done in such a way that all primary virtues of the linear trend to be preserved. For that purpose we offer to change (9) in the following manner

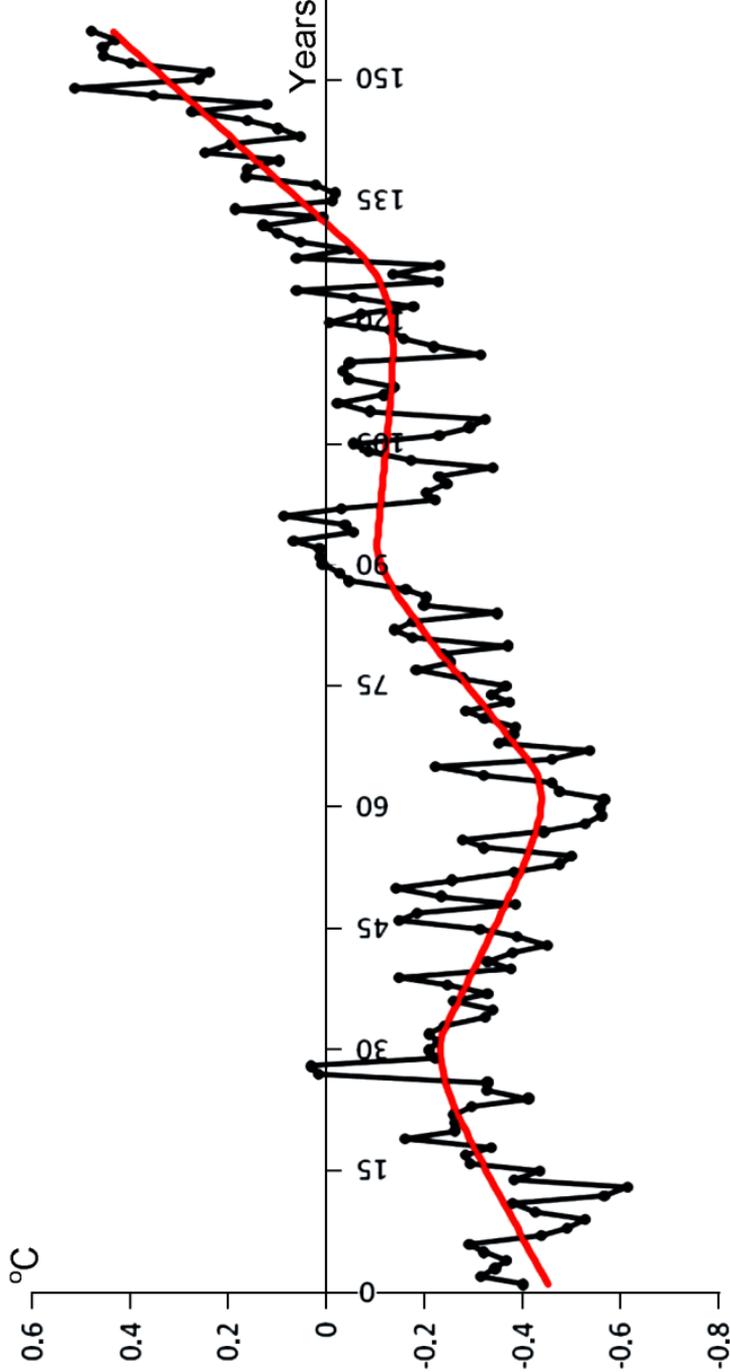


Figure 6. Graph of nonlinear trend approximation of yearly average temperature variations.

$$u_i = \bar{u}_{0i} + X_i(D_i)(t - t_{0i}) + \frac{X_i^{(nl)}(D_i, D_{i+1})}{7T_i^6} \cdot (t - t_{0i})^7 + \tilde{u}_i = \bar{u}_i + \tilde{u}_i \quad (10)$$

It follows from (10) that almost in all intervals T_i the values of (9) and (10) are close except for the field near t_{0i+1} . In this field the nonlinear seventh degree term is significant enough and allows to perform a smooth junction from X_i slope ratio to X_{i+1} . Let the values of X_i factors be the same that in the preceding section in the case of a linear trend. \bar{u}_{0i} and $X_i^{(nl)}$ factors will be calculated from the condition for continuity and smoothness, \bar{u}_i . From this the following conditions follow

$$\begin{aligned} \bar{u}_{0i+1} &= \bar{u}_{0i} + X_i T_i + \frac{1}{7} \cdot X_i^{(nl)} T_i \\ X_{i+1} &= X_i + X_i^{(nl)} \end{aligned} \quad (11)$$

In this respect the values of \bar{u}_{0i} and $X_i^{(nl)}$ are to be determined from best possible fit with the experimental results according to the least square method. The resultant numeric values of $X_i^{(nl)}$ are shown in Table 2.

The nonlinear trend approximation of the experimental results (10) is given in Figure 6.

Values $X_i^{(nl)}$ have turned out to be the values of the same order as the values of the linear trend factors X_i , including the sign-changing behavior. This points to the trend behavior has been changing since all time intervals T_i started.

Forecast for global temperature linear trend dynamics u.

The periods of global temperature linear trend variations T_i have turned out to be the values of the order of 30 – 31 years. This value is close to the tripled period of 11-year period of solar activity. There is no any reliable proof of relation of these processes in view.

In this model the forecast periods $i = 6, 7$ must have the following values: $T_6=31$ years, $T_7=30$ years. On the supposition of $X_6 = X_4$ and $X_7 = X_5$, which is equivalent of maintenance of a trend for last two periods $i = 4, 5$, we find out $\Delta\bar{u}_6 \approx -0.0016 \cdot 31^\circ\text{C} = -0.05^\circ\text{C}$ and $\Delta\bar{u}_7 \approx 0.0183 \cdot 30^\circ\text{C} = 0.55^\circ\text{C}$. By summing the resultant values up we find out $\Delta\bar{u}_{6,7} = 0.50^\circ\text{C}$, that is in 61 years the average global temperature should increase by 0.50°C . This is an added reason for the global warming trend.

A somewhat different forecast follows from the non-

linear trend model. According to this model, $X_6 = X_5 + X_5^{(nl)} = 0.0013^\circ\text{C}$, and thus we have $\Delta\bar{u}_6 = 0.040^\circ\text{C}$. Instead of temperature drop of 0.05°C , we have a slight 0.04°C growth for average global temperature in 31 years or in 2042.

Conclusion

The analysis and the forecast for yearly average temperature change we have carried out within the framework of the multifractal dynamics, show that the global warming trend should continue for the coming 60 years. Over this period the global temperature linear trend should rise by about 0.5°C .

References

- Jones P. D., T. M. L. Wigley (1990), Global Warming Trends, *Scientific American*, 263, 2, 84–91.
- Mandelbrot B. B. (1982), *The Fractal geometry of nature*, San Francisco, 460 pp.
- Brohan P., J. J. Kennedy, I. Haris, S. F. B. Tett and P. D. Jones (2006), Uncertainty estimates in regional and global observed

temperature changes: a new dataset from 1850, *J. Geophysical Research*, 111, D12106, doi:10.1029/2005JD006548.

Kudinov A. N., V. P. Tsvetkov, and I. V. Tsvetkov (2011), Catastrophes in the Multi-Fractal Dynamics of Social-Economic Systems, *Russian Journal of Mathematical Physics*, 18, 2, 149–155.

Kudinov A. N., V. P. Tsvetkov, and I. V. Tsvetkov (2009), Currency crisis and bifurcation phenomena in the framework of the fractal theory, (in Russian) *Finansy i kredit*, 46(326) 5–9.
